18.100A Practice problems for Chapter 1-11

The midterm exam will take place on March 22th Thursday 9:35-10:50.

As an open book exam, during the exam you can see
1. the textbook : *Introduction to Real Analysis* by A. Mattuck,
2. notes, copies, and scratch papers (at most 500 sheets of paper).

However, the following are NOT allowed to use
1. electronic devices.
2. the other books except the textbook.

When you write the poofs of problems, you can cite Theorems, Properties, and examples with proofs in the textbook Chapter 1-11. Moreover, a sheet of facts will be given and you can cite them.

However, you can not use exercises and problems in the textbook as well as problem sets, practice problems, and their solutions. If you have copies of the solutions and want to use them, please rewrite the proofs.

The problems in this file will be continuously updated until March 16th without announcements. The solutions will be provided in a separate file.

The problems with stars are challenging.

Problem 1. *Determine whether the following statements are true or false. If true then prove it, and if false then provide a counterexample.*

(1) *Suppose* $a_n > M$ *for* $n \gg 1$ *and* $\lim a_n = L$. *Then,* $L > M$.
(2) *Suppose* $\lim a_n^2 = L$. *Then,* $\lim a_n = \sqrt{L}$.
(3) *Suppose* $\{a_n b_n\}$ *and* $\{a_n\}$ *converge. Then,* b_n *also converges.*
(4) *Suppose* $\{a_n\}$ *and* $\{b_n\}$ *with* $b_n \neq 0$ *are bounded. Then,* a_n/b_n *is also bounded.*
(5) *Suppose a non-empty set* S *has its supremum. Then, the set* $S^2 = \{s^2 : s \in S\}$ *has its supremum and* $(\sup S)^2 = \sup S^2$.
(6) *A sequence of open intervals* $I_n = (a_n, b_n)$ *satisfies* $I_{n+1} \subset I_n$ *and* $\lim |b_n - a_n| = 0$. *Then, there exists a number* L *such that* $\lim a_n = \lim b_n = L$ *and* $L \in I_n$.
(7) *If* $\lim a_n = M$, *then* $\lim |a_n| = |M|$.
(8*) *A sequence* $\{n^2 a_n\}$ *converges. Then, the series* $\sum a_n$ *converges.*
(9) *Let* a_n *and* b_n *be Cauchy sequences. Then,* $a_n b_n$ *is also a Cauchy sequence.*
(10) *Let* $a_n > 0$ *be a Cauchy sequence. Then,* $\frac{1}{a_n}$ *is also a Cauchy sequence.*
(11) *Let* $f(x)$ *be defined for* $x \approx x_0$ *and* $\lim_{x \to x_0} f(x) = L$. *Then,* $L = f(x_0)$.
(12) *Let* $f(x)$ *be a bounded function defined for* $x \approx 0$. *Then,* $xf(x)$ *is continuous at* 0.

(13) *Suppose that $f(x)$ has both of the right- and left- hand limits. Then, $f(x)$ has the limit at x_0.*
(14) *Let $f(x)$ be an increasing function defined on an interval $[0,1]$. Then, $f(x)$ is left-continuous at 1.*
(15*) *Let $f(x)$ be an increasing function defined on an interval $[0,1]$. Then, $f(x)$ has the left-hand limit at 1.*

Problem 2. *Determine whether the following sequences are convergent or divergent. If convergent, find the limit and explain why it is the limit. If divergent, explain why the sequence is not convergent.*

$$(1)\, a_n = \frac{(-1)^n n}{2n+1} \qquad (2)\, a_n = \frac{n^3}{3^n} \qquad (3)\, a_n = \frac{2^n+1}{3^n+n^3} \qquad (4^*)\, a_n = \frac{n!}{n^n}$$

$$(5) a_{n+1} = \left(\frac{a_n}{2}\right)^2,\ a_0 < 4, \qquad (6) a_{n+1} = \left(\frac{a_n}{2}\right)^2,\ a_0 > 4.$$

$$(7^*) 4a_{n+1} = 5 - a_n^2,\ 0 < a_0 < 2, \qquad (8) a_{n+1} = \sqrt{2a_n - 1},\ a_0 > 1.$$

Fact needed for (4^*) : $\lim(1+\frac{1}{n})^n = e \approx 2.71828.... > 1$.

Problem 3. *Let $a_{n+1} = \frac{2}{1+a_n}$ and $a_0 > 1$.*
(1) *Show that the subsequence of even terms a_{2n} is decreasing and bounded below, and the subsequence of odd terms a_{2n-1} is increasing and bounded above.*
(2) *Show the convergence of a_n, and fine the limit.*

Problem 4. *Let $a_{n+1} = \frac{1}{2+a_n}$ and $a_0 > 0$.*
(1) *Show that $\{a_n\}$ is a Cauchy sequence.*
(2) *Find the limit of $\{a_n\}$ and explain why it is the limit.*

Problem 5. *Let S, T be non-empty sets bounded above. Suppose $s, t > 0$ holds for all $s \in S$ and $t \in T$. Then, we have $(\sup S)(\sup T) = \sup ST$, where $ST = \{st : s \in S, t \in T\}$.*

Problem 6. *Determine whether the following series are convergent or divergent, and explain why they are convergent or divergent.*

$$(1) \sum_{n=1}^{\infty} \frac{n}{2n+1} \qquad (2) \sum_{n=1}^{\infty} \frac{n^3}{3^n} \qquad (3) \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \qquad (4) \sum_{n=1}^{\infty} \frac{2n}{n^2+1}.$$

Problem 7. *Find the radius of convergence of the following power series, and explain why.*

$$(1)\ \sum_{n=0}^{\infty} \frac{x^{2n}}{3^n} \qquad\qquad (2)\ \sum_{n=1}^{\infty} \frac{x^n}{2^n(n+1)}.$$

Problem 8 (8*). *Let $f(x)$ be a continuous function defined on $\mathbb{R}$. Suppose that $f(m2^{-n}) \geq 0$ holds for all integer $m \in \mathbb{Z}$ and natural number $n \in \mathbb{N}$. Show that $f(x) \geq 0$ holds for all $x \in \mathbb{R}$.*

Problem 9. *Let $f(x)$ be a bounded function defined on $\mathbb{R}$, and let $F(x) = \int_0^x f(t)dt$. Show that $F(x)$ is continuous on $\mathbb{R}$.*

Problem 10. *Suppose that $f(x)$ is a continuous function defined on $\mathbb{R}$, and $f(x) \geq 0$ holds for all $x \in \mathbb{R}$. Show that $g(x) = \sqrt{f(x)}$ is continuous on $\mathbb{R}$.*

Problem 11. *Suppose that a continuous function $f(x)$ is defined on $[a,b]$ with $a \neq b$, and $f(x)$ is strictly increasing on (a,b). Show that $f(x)$ is strictly increasing on $[a,b]$.*

Problem 12. *Suppose that $f(x), g(x)$ are continuous functions defined on $\mathbb{R}$. Show that the function $h(x) = \max\{f(x), g(x)\}$ is continuous on $\mathbb{R}$.*

Sample Exam

Try to solve the following problems in 75 minutes. If you score is greater than 100, you will receive 100 points.

1.(20 points) Determine whether the following statements are true or false. If true then prove it, and if false then provide a counterexample.

(1) Suppose $\{a_n\}$ and $\{b_n\}$ with $b_n \neq 0$ are bounded. Then, a_n/b_n is also bounded.
(2) Let $f(x)$ be a bounded function defined for $x \approx 0$. Then, $xf(x)$ is continuous at 0.

2.(25 points) Let $a_{n+1} = \sqrt{2a_n - 1}$ and $a_0 > 1$. Determine whether it is convergent or divergent. If convergent, find the limit and explain why it is the limit. If divergent, explain why the sequence is not convergent.

3.(25 points) Let $a_{n+1} = \frac{1}{2+a_n}$ and $a_0 > 0$. Show that $\{a_n\}$ is a Cauchy sequence.

4.(10 points) Find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{2^n(n+1)}$, and explain why.

5.(20 points) Suppose that $f(x)$ is a continuous function defined on $\mathbb{R}$, and $f(x) \geq 0$ holds for all $x \in \mathbb{R}$. Show that $g(x) = \sqrt{f(x)}$ is continuous on $\mathbb{R}$.

6.(10 points, bonus problem) Find a function $f(x)$ satisfying the following conditions

(1) $f(x)$ is continuous on $(0, +\infty)$ and has the right-hand limit at 0.
(2) Its derivative $f'(x)$ is continuous on $(0, +\infty)$.
(3) $f'(x)$ does not have the right-hand limit at 0.

7.(10 points, bonus problem) Show that

$$4 = \sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$